# ON ASYMPTOTIC STABILITY OF SYSTEMS WITH AFTER-EFFECT CONTAINING A SMALL PARAMETER AS COEFFICIENT OF THE DERIVATIVES

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The author considers systems of differential equations with after-effect that contain a small parameter as a coefficient of the derivatives. On the basis of the given uniform asymptotic stability of the degenerate system of differential equations, and of the uniform stepwise asymptotic stability of a certain auxiliary system, he establishes the uniform asymptotic stability of the original system. The problems are treated by the Liapunov-Chetaev [1,2] method developed for systems of equations with after-effect by Krasovskii [3]. It is mentioned that problems on the stability of systems of differential equations with after-effect containing a small parameter were considered by El'sgol'ts [4]. The present work extends certain results of Krasovskii [5] to systems with after-effect.

**1. Linear systems.** We consider a system of differential equations with after-effect of the form

$$\frac{dx_i}{dt} = \sum_{l=1}^{m} a_{il}(t)\, x_l + \sum_{s=1}^{n} b_{is}(t)\, y_s + \sum_{l=1}^{m} \alpha_{il}(t)\, x_l(t-\theta)$$

$$\mu \frac{dy_j}{dt} = \sum_{l=1}^{m} c_{jl}(t)\, x_l + \sum_{s=1}^{n} d_{js}(t)\, y_s + \sum_{l=1}^{m} \beta_{jl}(t)\, x_l(t-\theta) \qquad \begin{pmatrix} i = 1, \ldots, m \\ j = 1, \ldots, n \end{pmatrix}$$

$$x_{i0} = g_{i0}(t) \quad \text{when } t_0 - \theta \leqslant t_i \leqslant t_0, \qquad y_{j0} = p_{j0} \tag{1.1}$$

where $\mu$ is a positive small parameter, and $\theta$ is the delay constant (time lag).

Let us assume that the coefficients $a_{il}(t)$, $b_{is}(t)$, $\alpha_{il}(t)$, $c_{jl}(t)$, $d_{js}(t)$, and $\beta_{jl}(t)$ are continuous bounded functions of the argument $t$ which have continuous bounded derivatives for values of $t$ such that $t_0 \leqslant t < \infty$. Furthermore, we assume that the following condition is satisfied,

$$\begin{vmatrix} d_{11}(t) \ldots d_{1n}(t) \\ \ldots\ldots\ldots \\ d_{n1}(t) \ldots d_{nn}(t) \end{vmatrix} > \beta > 0 \tag{1.2}$$

The degenerate system of the original system has the form

$$\frac{dx_i}{dt} = \sum_{l=1}^{m} a_{il}(t)\, x_l + \sum_{s=1}^{n} b_{is}(t)\, y_s + \sum_{l=1}^{m} \alpha_{il}(t)\, x_l(t-\theta) \qquad (i = 1, \ldots, m) \tag{1.3}$$

$$\sum_{l=1}^{m} c_{jl}(t)\, x_l + \sum_{s=1}^{n} d_{js}(t)\, y_s + \sum_{l=1}^{m} \beta_{jl}(t)\, x_l(t-\theta) = 0 \qquad (j = 1, \ldots, n) \tag{1.4}$$

$$x_{i0} = g_{i0}(t) \quad \text{when } t_0 - \theta \leqslant t \leqslant t_0$$

We denote by $x_i = x_i(t, \mu)$ and $y_j = y_j(t, \mu)$ the solution of the system (1.1), and by the symbols $x_i = x_i(t)$, and $y_j = y_j(t)$ the solution of the degenerate system (1.3), (1.4).

The solution of the system of $n$ linear algebraic Equations (1.4) for $y_1, \ldots, y_n$ has the form

$$y_s = \sum_{l=1}^{m} \lambda_{sl}(t)\, x_l(t) + \sum_{l=1}^{m} \gamma_{sl}(t)\, x_l(t-\theta) \qquad (s = 1, \ldots, n) \tag{1.5}$$

Here $\lambda_{sl}(t)$ and $\gamma_{sl}(t)$ are bounded continuous functions of $t$.

The substitution of the Expressions (1.5) into the first $m$ Equations (1.3) of the degenerate system leads to a system of $m$ differential equations with after-effect

$$\frac{dx_i}{dt} = \sum_{k=1}^{m} r_{ik}(t)\, x_k(t) + \sum_{k=1}^{m} \tau_{ik}(t)\, x_k(t-\theta) \tag{1.6}$$

Here $r_{ik}(t)$ and $\tau_{ik}(t)$ are continuous bounded functions of $t$.

We will call the system

$$\frac{dy_j}{dt} = \sum_{s=1}^{n} d_{js}(t)\, y_s \qquad (j = 1, \ldots, n) \tag{1.7}$$

of $n$ linear differential equations with variable coefficients, the auxiliary system of equations.

We shall show in the sequel that for sufficiently small values of the parameter $\mu$, and under certain conditions, the trajectory of the original system (1.1) tends to the trajectory of the degenerate system of Equations (1.3) and (1.4), and we shall show also that the stability of the auxiliary systems implies the stability of the solution of the original system (1.1).

*Theorem 1.1.* Suppose that the following conditions hold for the systems of equations with after-effect (1.1) and for the corresponding degenerate system (1.3), (1.4):

a) the system of differential equations with after-effect (1.6) is uniformly asymptotically stable;

b) for every fixed value of $\omega$ the systems of equations

$$\frac{dy_j}{dt} = \sum_{s=1}^{n} d_{js}(\omega)\, y_s \qquad (j = 1, \ldots, n) \tag{1.8}$$

with constant coefficients are asymptotically stable, uniformly in $\omega \ni [t_0, \infty]$ (or, what is the same thing, the roots of the characteristic equation $|d_{js} - \rho\,\delta_{js}| = 0$ satisfy the condition $\mathrm{Re}\,\rho < -\Delta$, $\Delta$ is a positive constant, $\delta_{js} = 0$ if $j \neq s$, and $\delta_{js} = 1$ if $j = s$).

Under these conditions the following assertions are valid.

1) For every given $\epsilon > 0$, there exists a number $\mu_0$ such that the following inequalities hold:

$$|x_i(t, \mu) - x_i(t)| < \varepsilon, \qquad |y_j(t, \mu) - y_j(t)| < \varepsilon \quad \text{when } t > t_1,\ \mu < \mu_0 \tag{1.9}$$

2) For sufficiently small $\mu_0$, the original system of differential equations with after-effect is asymptotically stable.

3) For a given solution of the degenerate system (1.3), (1.4) and for every $Q > 0$, there exists a sufficiently small $\mu_0$ such that the number $t_1$ of the condition (1.9) differs from the number $t_0$ by less than a given number $\delta > 0$ for all initial conditions $|y_j(t_0, \mu) - y_0(t_0)| < Q$.

*Proof.* We introduce new variables $\xi_i$ and $\eta_i$ by means of the equations

$$\xi_i(t, \mu) = x_i(t, \mu) - x_i(t)$$

$$\eta_j(t, \mu) = y_j(t, \mu) - \sum_{s=1}^{m} \lambda_{js}(t)\, x_s(t, \mu) - \sum_{s=1}^{m} \gamma_{jl}(t)\, x_l(t - \theta, \mu)$$

Let us construct the differential equations of the disturbed motion which will be satisfied by the variables $\xi_i$ and $\eta_j$. These equations must be constructed separately for values of $t$ between $t_0$ and $t_0 + \theta$, and for

$t$ greater than $t_0 + \theta$.

$$\frac{d\xi_i(t,\mu)}{dt} = \frac{dx_i(t,\mu)}{dt} - \frac{dx_i(t)}{dt}$$

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{dy_j(t,\mu)}{dt} - \sum_{s=1}^{m} \frac{d\lambda_{js}(t)}{dt} x_s(t,\mu) - \sum_{s=1}^{m} \lambda_{js}(t) \frac{dx_s(t,\mu)}{dt} -$$

$$- \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt} x_l(t-\theta,\mu) - \sum_{l=1}^{m} \gamma_{jl}(t) \frac{dx_l(t-\theta,\mu)}{dt}$$

On the basis of these equations we have

a) when $t_0 \leqslant t \leqslant t_0 + \theta$

$$\frac{d\xi_i(t,\mu)}{dt} = \sum_{l=1}^{m} r_{il}(t)\,\xi_l(t,\mu) + \sum_{l=1}^{m} \tau_{il}(t)\,\xi_l(t-\theta,\mu) - \sum_{s=1}^{n} b_{is}(t)\,\eta_s(t,\mu)$$

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\eta_s(t,\mu) - \sum_{k=1}^{m} \lambda_{jk}(t)\Big[\sum_{l=1}^{m} r_{kl}(t)\,\xi_l(t,\mu) +$$

$$+ \sum_{l=1}^{m} \tau_{kl}(t)\,\xi_l(t-\theta,\mu)\Big] - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt}\xi_k(t,\mu) - \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt}\xi_l(t-\theta,\mu) -$$

$$- \sum_{k=1}^{m} \lambda_{jk}(t)\Big[\sum_{f=1}^{m} r_{kf}(t)\,x_f(t) + \sum_{f=1}^{m} \tau_{kf}(t)\,x_f(t-\theta)\Big] - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt} x_k(t) -$$

$$- \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt} x_l(t-\theta) - \sum_{l=1}^{m} \gamma_{jl}(t)\frac{dg_{l0}(t)}{dt} - \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{s=1}^{n} b_{ks}(t)\,\eta_s(t,\mu) \qquad (1.10)$$

b) when $t > t_0 + \theta$

$$\frac{d\xi_i(t,\mu)}{dt} = \sum_{l=1}^{m} r_{il}(t)\,\xi_l(t,\mu) + \sum_{l=1}^{m} \tau_{il}(t)\,\xi_l(t-\theta,\mu) + \sum_{s=1}^{n} b_{is}(t)\,\eta_s(t,\mu)$$

$$\frac{d\eta_j(t,\mu)}{dt} = \frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\eta_s(t,\mu) - \sum_{k=1}^{m} \lambda_{jk}(t)\Big[\sum_{l=1}^{m} r_{kl}(t)\,\xi_l(t,\mu) +$$

$$+ \sum_{l=1}^{m} \tau_{kl}(t)\,\xi_l(t-\theta,\mu)\Big] - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt}\xi_k(t,\mu) - \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt}\xi_l(t-\theta,\mu) -$$

$$- \sum_{k=1}^{m} \lambda_{jk}(t)\Big[\sum_{f=1}^{m} r_{kf}(t)\,x_f(t) + \sum_{f=1}^{m} \tau_{kf}(t)\,x_f(t-\theta)\Big] - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt} x_k(t) -$$

$$- \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt} x_l(t-\theta) - \sum_{l=1}^{m} \gamma_{jl}(t)\Big[\sum_{f=1}^{m} r_{lf}(t)\,\xi_f(t-\theta,\mu) +$$

$$+ \sum_{f=1}^{m} \tau_{lf}(t)\,\xi_f(t-2\theta,\mu)\Big] - \sum_{l=1}^{m} \gamma_{jl}(t)\Big[\sum_{k=1}^{m} r_{lk}(t)\,x_k(t-\theta) + \sum_{k=1}^{m} \tau_{lk}(t)\,x_k(t-2\theta)\Big] -$$

$$- \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{s=1}^{n} b_{ks}(t)\,\eta_s(t,\mu) - \sum_{l=1}^{m} \gamma_{jl}(t) \sum_{s=1}^{n} b_{ls}(t)\,\eta_s(t-\theta,\mu) \qquad (1.11)$$

For the proof of the theorem one must show that the solutions $\xi_i(t,\mu)$, $\eta_j(t, \mu)$ of the systems (1.10) and (1.11) satisfy the following conditions: for every given set of numbers $Q$, $\epsilon$ and $\delta$ there exists a number $\mu_0 > 0$ such that for the given initial values $\xi_{i0} = 0$ when $t_0 - \theta \leqslant t \leqslant t_0$, $|\eta_{ij}| < Q$, it is true that $|\xi_i(t, \mu)| < \epsilon$, $|\eta_j(t, \mu)| < \epsilon$ when $t > t_0 + \delta$, provided $\mu < \mu_0$.

By the hypotheses of the theorem, the system of differential equations with after-effect (1.6) is uniformly asymptotically stable. Hence there exists for it a positive definite functional $v[\xi_1(\vartheta), \xi_2(\vartheta), \ldots, \xi_m(\vartheta), t] = v[\xi_i(\vartheta), t]$ for which the following inequality is valid

$$c_1 \| \xi_i(\vartheta) \|^2 \leqslant v[\xi_i(\vartheta), t] \leqslant c_2 \| \xi_i(\vartheta) \|^2 \tag{1.12}$$

and such that its derivative, in view of (1.6), satisfies the inequality

$$\frac{dv[\xi_i(\vartheta), t]}{dt} \leqslant c_3 \| \xi_i(\vartheta) \|^2 \tag{1.13}$$

Furthermore, we have the inequality

$$\overline{\lim} \frac{| v[\xi_{i1}(\vartheta), t] - v[\xi_{i2}(\vartheta), t] |}{\| \xi_{i1}(\vartheta) - \xi_{i2}(\vartheta) \|} = c_4 \| \xi_i(\vartheta) \| \tag{1.14}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are positive numbers, and [3 (Section 33)]

$$\| \xi_i(\vartheta) \| = \sup_\vartheta \sqrt{\xi_1^2(\vartheta) + \ldots + \xi_m^2(\vartheta)} \quad \text{for } t_0 - 2\theta \leqslant \vartheta \leqslant t_0$$

We note that in [3 (Section 33)], there is established the existence of a functional $v[x_i(\vartheta), t]$ satisfying the inequalities

$$c_1 \| x(\vartheta) \| \leqslant v[x(\vartheta), t] \leqslant c_2 \| x(\vartheta) \|, \qquad \limsup \left( \frac{\Delta v}{\Delta t} \right)_{(33.1)} \leqslant c_3 \| x(\vartheta) \|$$
$$| v[x''(\vartheta), t] - v[x'(\vartheta), t] | \leqslant c_4 \| x''(\vartheta) - x'(\vartheta) \|,$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are positive numbers, and $\| x(\vartheta) \| = \sup | x_i(\vartheta) |$ when $-\theta \leqslant \vartheta \leqslant 0$. By considerations similar to those of [3], one can establish the existence of a functional satisfying the inequalities given here.

From the second condition of the theorem it follows that the system (1.7) is asymptotically stable, uniformly in $\omega$, for all values of $\omega$ and hence, there exists, for the system (1.7), a positive definite quadratic form $w(\omega, \eta_1, \ldots, \eta_n) = w(\omega, \eta_j)$ whose total derivative is negative definite in view of the system (1.7) (when $\omega = \text{const}$). Hereby the partial derivatives $\partial w/\partial \omega$ will be bounded and the inequalities of the positive definiteness and negative definiteness of $w(\omega, \eta_i)$ and $(dw/dt)_{(1.7)}$, respectively, will hold uniformly in $\omega$.

Let us construct a positive definite functional of the form

$$u\,[t,\ \xi_i(\vartheta),\ \eta_j] = v\,[\xi_i(\vartheta),\ t] + w\,(t,\ \eta_j) \tag{1.15}$$

We shall show that for small enough $\mu$ the functional satisfies the conditions of the Theorem (31.4) of [3 (p.195)] in the region $\|\xi_i\| > M_1$, $\|\eta_j\| > M_1$.

The total derivative of the functional $u[t,\ \xi_i(\vartheta),\ \eta_i]$ has, in view of (1.10), the form

$$\begin{aligned}\Big(\frac{du\,[t,\ \xi_i(\vartheta),\ \eta_j]}{dt}\Big)_{(1.10)} &= \Big(\frac{dv\,[\xi_i(\vartheta),\ t]}{dt}\Big)_{(1.7)} + F_1(t,\ \xi_i,\ \eta_j) + \frac{\partial w(t,\ \eta_j)}{\partial t} + \\ &+ \sum_{j=1}^{n} \frac{\partial w(t,\ \eta_j)}{\partial \eta_j}\Big\{\frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\eta_s(t,\ \mu) - \sum_{k=1}^{m}\lambda_{jk}(t)\Big[\sum_{l=1}^{m} r_{kl}(t)\,\xi_l(t,\ \mu) + \\ &+ \sum_{l=1}^{m}\tau_{kl}(t)\,\xi_l(t-\theta,\ \mu)\Big] - \sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}\,\xi_k(t,\ \mu) - \sum_{l=1}^{m}\frac{d\gamma_{jl}(t)}{dt}\,\xi_l(t-\theta,\ \mu) - \\ &- \sum_{k=1}^{m}\lambda_{jk}(t)\Big[\sum_{f=1}^{m} r_{kf}(t)\,x_f(t) + \sum_{f=1}^{m}\tau_{kf}(t)\,x_f(t-\theta)\Big] - \sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}\,x_k(t) - \\ &- \sum_{l=1}^{m}\frac{d\gamma_{jl}(t)}{dt}\,x_l(t-\theta) - \sum_{l=1}^{m}\gamma_{jl}(t)\,\frac{dg_{l0}(t)}{dt} - \sum_{k=1}^{m}\lambda_{jk}(t)\sum_{s=1}^{n} b_{ks}(t)\,\eta_s(t,\ \mu)\Big\}\end{aligned} \tag{1.16}$$

Here, the symbol $F_1(t,\ \xi_i,\ \eta_j)$ denotes all terms of the derivative of the functional $v[\xi_i(\vartheta),\ t]$, found by means of the first $m$ equations of the disturbed motion (1.10), that contain as factors the quantities $b_{i1}(t)\eta_1(t,\ \mu) + \ldots + b_{in}(t)\eta_n(t,\ \mu)$.

Keeping in mind the properties of the functional $v[\xi_i(\vartheta),\ t]$ and of the function $w(t,\ \eta_j)$, one can derive from (1.16) the inequality (1.17)

$$\begin{aligned}\Big(\frac{du\,[t,\ \xi_i(\vartheta),\ \eta_j]}{dt}\Big)_{(1.10)} &\leqslant -c_3\|\xi_i\|^2 + M^* c_4\|\xi_i\|\Big(\sum_{s=1}^{n}\eta_s^2(t)\Big)^{1/2} + \frac{1}{\mu}\Big[-\sum_{s=1}^{n}\eta_s^2(t)\Big] + \\ &+ N_1\|\xi_i\|\Big(\sum_{s=1}^{n}\eta_s^2(t)\Big)^{1/2} + N_2\Big(\sum_{s=1}^{n}\eta_s^2(t)\Big)^{1/2} + N_3\sum_{s=1}^{n}\eta_s^2(t)\end{aligned} \tag{1.17}$$

where $M^*$, $N_1$, $N_2$ and $N_3$ are some numbers. The right-hand side of this inequality (1.17) is a quadratic form in $\|\xi_i\| = \rho$, and $\|\eta_j\| = \sigma$ if one drops in it the next to the last term.

For sufficiently small $\mu$, this form is negative definite. Hence, for such $\mu$, the left-hand side of the inequality (1.17) is less than zero outside a neighborhood of the point $\xi_i = 0$, $\eta_j = 0$. This reasoning is analogous to that found in [5].

In view of the system (1.11), the total derivative of the functional $u[t, \xi_i(\theta), \eta_j]$ will, for $t > t_0 + \theta$, have the form

$$\left(\frac{du\,[t,\ \xi_i(\theta),\ \eta_j]}{dt}\right)_{(1.11)} = \left[\frac{dv\,[\xi_i(\theta),\ t]}{dt}\right]_{(1.6,\xi)} + F_1(t,\ \xi_i,\ \eta_j) + \frac{\partial w(t,\ \eta_j)}{\partial t} +$$
$$+\sum_{j=1}^{n}\frac{\partial w(t,\ \eta_j)}{\partial \eta_j}\left\{\frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\eta_s(t,\ \mu) - \sum_{k=1}^{m}\lambda_{jk}(t)\left[\sum_{l=1}^{m} r_{kl}(t)\,\xi_l(t,\ \mu) + \right.\right.$$
$$\left. + \sum_{l=1}^{m}\tau_{kl}(t)\,\xi_l(t-\theta,\ \mu)\right] - \sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}\,\xi_k(t,\ \mu) - \sum_{l=1}^{m}\frac{d\gamma_{jl}(t)}{dt}\,\xi_l(t-\theta,\ \mu) -$$
$$-\sum_{k=1}^{m}\lambda_{jk}(t)\left[\sum_{f=1}^{m} r_{kf}(t)\,x_f(t) + \sum_{f=1}^{m}\tau_{kf}(t)\,x_f(t-\theta)\right] - \sum_{k=1}^{m}\frac{d\lambda_{jk}(t)}{dt}\,x_k(t) -$$
$$-\sum_{l=1}^{m}\frac{d\gamma_{jl}(t)}{dt}x_l(t-\theta) - \sum_{l=1}^{m}\gamma_{jl}(t)\left[\sum_{f=1}^{m} r_{lf}(t)\,\xi_f(t-\theta,\ \mu) + \sum_{f=1}^{m}\tau_{lf}(t)\,\xi_f(t-2\theta,\ \mu)\right] -$$
$$-\sum_{l=1}^{m}\gamma_{jl}(t)\left[\sum_{k=1}^{m} r_{lk}(t)\,x_k(t-\theta) + \sum_{k=1}^{m}\tau_{lk}(t)\,x_k(t-2\theta)\right] -$$
$$\left. -\sum_{l=1}^{m}\lambda_{jk}(t)\sum_{s=1}^{n} b_{ks}(t)\,\eta_s(t,\ \mu) - \sum_{l=1}^{m}\gamma_{jl}(t)\sum_{s=1}^{n} b_{ls}(t)\,\eta_s(t-\theta,\ \mu)\right\} \tag{1.18}$$

The symbol $F_1(t,\ \xi_i,\ \eta_j)$ denotes here the terms of the derivative $v[\xi_i(\theta),\ t]$ which contain as factors the quantities $b_{i1}(t)\eta_1(t,\ \mu) + \ldots + b_{in}(t)\eta_n(t,\ \mu)$.

From the Equation (1.18) one can deduce the following inequality

$$\left(\frac{du\,[t,\ \xi_i(\theta),\ \eta_j]}{dt}\right)_{(1.11)} \leqslant -c_3\|\xi_i\|^2 + M^* c_4\|\xi_i\|\left(\sum_{s=1}^{n}\eta_s^2(t)\right)^{1/2} +$$
$$+\frac{1}{\mu}\left[-\sum_{s=1}^{n}\eta_s^2(t)\right] + N_1\|\xi_i\|\left(\sum_{s=1}^{n}\eta_s^2(t)\right)^{1/2} + N_2\left(\sum_{s=1}^{n}\eta_s^2(t)\right)^{1/2} +$$
$$+ N_3\sum_{s=1}^{n}\eta_s^2(t) + N_4\left(\sum_{s=1}^{n}\eta_s^2(t)\right)^{1/2}\left(\sum_{s=1}^{n}\eta_s^2(t-\theta)\right)^{1/2} \tag{1.19}$$

where $M^*$, $N_1$, $N_2$, $N_3$ and $N_4$ are some numbers.

Let us consider the curves

$$\sum_{s=1}^{n}\eta_s^2(t-\theta) < g\sum_{s=1}^{n}\eta_s^2(t) \tag{1.20}$$

where $g$ is some number. On these curves, the right-hand side of (1.19), with the exclusion of the one term containing $N_2$, can be treated as a quadratic form in $\|\xi_i\| = \rho$, and $\|\eta_j\| = \sigma$. From this it follows that for sufficiently small $\mu$ the left side of the inequality (1.19) without the linear term, will (on the curves 1.20) be less than some negative definite form in $\rho$ and $\sigma$. This means that the functional $u[t, \xi_i(\theta), \eta_j]$ satisfies all the conditions of the Theorem (31.4) of [3] when $|\rho| > M_1$, $|\sigma| > M_1$, where $M_1$ is some number.

One can show that as $\mu$ tends to zero the region $|\rho| > M_1$, $|\sigma| > M$, outside of which the left side of the inequality (1.19) on the curves (1.20) is negative definite, also tends to zero, i.e. $M_1 \to 0$ when $\mu \to 0$. In consequence of this, the region $M < |\xi_i|$, $M < |\eta_j|$ also goes to zero.

Repeating now the arguments of the proof of the Theorem (31.4) of [3 (p.185)], one can establish the validity of the first assertion of the theorem.

Next, we verify the asymptotic stability of the system (1.1). For this purpose we consider two solutions of this system which differ in their initial conditions, or two solutions of the system (1.10) with different initial conditions. We will denote these solutions by the symbols $\xi_{i1}(t, \mu)$, $\eta_{j1}(t, \mu)$, and $\xi_{i2}(t, \mu)$, $\eta_{j2}(t, \mu)$.

Let us construct the differential equations with after-effect which must be satisfied by the differences of these solutions:

$$\alpha_i(t, \mu) = \xi_{i1}(t, \mu) - \xi_{i2}(t, \mu), \qquad \beta_j(t, \mu) = \eta_{j1}(t, \mu) - \eta_{j2}(t, \mu) \tag{1.21}$$

On the basis of the Equations (1.21) and (1.10) we have

$$\frac{d\alpha_i(t, \mu)}{dt} = \sum_{l=1}^{m} r_{il}(t)\,\alpha_l(t, \mu) + \sum_{l=1}^{m} \tau_{il}(t)\,\alpha_l(t-\theta, \mu) + \sum_{s=1}^{n} b_{is}(t)\,\beta_s(t, \mu)$$

$$\frac{d\beta_j(t, \mu)}{dt} = \frac{1}{\mu}\sum_{s=1}^{n} d_{js}(t)\,\beta_s(t, \mu) - \sum_{k=1}^{m} \lambda_{jk}(t)\Big[\sum_{l=1}^{m} r_{kl}(t)\,\alpha_l(t, \mu) +$$

$$+ \sum_{l=1}^{m} \tau_{kl}(t)\,\alpha_l(t-\theta, \mu)\Big] - \sum_{k=1}^{m} \frac{d\lambda_{jk}(t)}{dt}\alpha_k(t, \mu) - \sum_{l=1}^{m} \frac{d\gamma_{jl}(t)}{dt}\alpha_l(t-\theta, \mu) -$$

$$- \sum_{k=1}^{m} \lambda_{jk}(t) \sum_{s=1}^{n} b_{ks}(t)\,\beta_s(t, \mu) \tag{1.22}$$

Let us consider the positive definite functional of the variables $t$, $\alpha_i$, and $\beta_j$ in the form

$$u[t, \alpha_i(\vartheta), \beta_j] = v[t, \alpha_i(\vartheta)] + w(t, \beta_j) \tag{1.23}$$

which was constructed as before (1.15). For sufficiently small values of the parameter $\mu$, this functional has a negative definite total derivative $d[t, \alpha_i(\vartheta), \beta_j]/dt$, evaluated by the use of (1.22). This can be verified in an analogous way as that used for the Equation (1.14). Hence, the functional (1.23) insures the asymptotic stability of the system of equations with after-effect (1.22). From the asymptotic stability of the system (1.22) follows the asymptotic stability of the original system of equations with after-effect (1.1).

It remains to show that as $\mu \to 0$, the quantity $\delta$ also goes to zero. Indeed, if $\mu$ is sufficiently small, the derivative $du[t, \xi_i(\vartheta), \eta_j]/dt$ will (in accordance with (1.16)) be a negative quantity, large in absolute value, provided $\rho$ remains small. But at the initial instant and when $t_0 - \theta \leqslant t \leqslant t_0$, the quantity $\rho = 0$, and it cannot increase much during the short period $t_0 \leqslant t \leqslant t_0 + \delta$, $(\delta > 0)$ because of its integrability.

But since $du[t, \xi_i(\vartheta), \eta_j]/dt$ is a negative quantity, large in absolute value, the positive definite functional $u[t, \xi_i(\vartheta), \eta_j]$ has to decrease very rapidly. This can occur, however, only under the condition that at a certain moment, $u[t, \xi_i(\vartheta), \eta_j]$ comes close to zero. This proves the theorem.

**2. Nonlinear systems.** Let us consider a system of nonlinear differential equations with after-effect

$$\frac{dx_i}{dt} = X_i[x_s, y_k, x_l(t-\theta), t], \qquad \mu \frac{dy_j}{dt} = Y_j[x_s, y_k, x_l(t-\theta), t] \tag{2.1}$$

$$x_{i0} = g_{i0}(t) \text{ when } t_0 - \theta \leqslant t \leqslant t_0, \qquad y_{j0} = b_{j0} \qquad \begin{pmatrix} i, s, l = 1, \ldots, m \\ j, k = 1, \ldots, n \end{pmatrix}$$

Here $\mu$ is a small positive parameter, $\theta$ is the constant time lag.

$$X_i[x_s, y_k, x_l(t-\theta), t] =$$
$$= X_i[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, x_1(t-\theta), x_2(t-\theta), \ldots, x_m(t-\theta), t]$$

$$Y_j[x_s, y_k, x_l(t-\theta), t] =$$
$$= Y_j[x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, x_1(t-\theta), x_2(t-\theta), \ldots, x_m(t-\theta), t]$$

Let us assume that the functions

$$X_i[x_s, y_k, x_l(t-\theta), t], \qquad Y_j[x_s, y_k, x_l(t-\theta), t]$$
$$(i, s, l = 1, \ldots, m; \quad j, k = 1, \ldots, n)$$

have continuous bounded derivatives with respect to all their arguments in the region $|x_s| \leqslant \infty$, $|y_k| \leqslant \infty$, $t_0 \leqslant t < \infty$, while

$$\frac{D(Y_1, Y_2, \ldots, Y_n)}{D(y_1, y_2, \ldots, y_n)} \neq 0$$

The degenerate system for the Equations (2.1), with $\mu = 0$, has the form

$$\frac{dx_i}{dt} = X_i[x_s, y_k, x_l(t-\theta), t], \qquad Y_j[x_s, y_k, x_l(t-\theta), t] = 0$$
$$x_{i0} = g_{i0}(t), \qquad t_0 - \theta \leqslant t \leqslant t_0 \tag{2.2}$$

Let us suppose that the system of $n$ equations $Y[x_s, y_k, x_l(t-\theta), t] = 0$ has a unique solution the set of functions $y_j = f_j[x_s, x_l(t-\theta), t]$ which have bounded partial derivatives with respect to all their arguments $(j = 1, \ldots, n)$.

Let us substitute $y_j = f_j[x_s, x_l(t-\theta), t]$ into the first $m$ Equations (2.2). We thus obtain

$$\frac{dx_i}{dt} = X_i[x_s, f_k(x_s, x_l(t-\theta), t), x_l(t-\theta), t] = F_i[x_s, x_l(t-\theta), t]$$
$$x_{i0} = g_{i0}(t) \text{ when } t_0 - \theta \leqslant t \leqslant t_0 \tag{2.3}$$

For the given initial condition, let us denote the solution of the original system (2.1) by $x_i = x_i(t, \mu)$, $y_j = y_j(t, \mu)$. For the corresponding initial conditions, we will denote the solution of the degenerate system of Equations (2.2) by $x_i = x_i(t)$, $y_j = y_j(t) = f_j[x_s(t), x_l(t-\theta), t]$.

Let us set up the differential equations of the disturbed motion for the given solution $x_i = x_i(t)$ of the system (2.3) by starting out with the equations $z_i(t) = x_i^*(t) - x_i(t)$, where $x_i^*(t)$ is the solution of the system (2.3) which corresponds to the change of the initial conditions $\Delta x_{i0} = x_{i0}^* - x_{i0}$. We thus obtain

$$\frac{dz_i(t)}{dt} = F_i[z_s(t) + x_s(t), z_l(t-\theta) + x_l(t-\theta), t] - F_i[x_s(t), x_l(t-\theta), t] =$$
$$= X_i\{z_s(t) + x_s(t), f_k[z_s(t) + x_s(t), z_l(t-\theta) + x_l(t-\theta), t], z_l(t-\theta) +$$
$$+ x_l(t-\theta), t\} - X_i\{x_s(t), f_k[x_s(t), x_l(t-\theta), t], x_l(t-\theta), t\} \tag{2.4}$$

Let us suppose that the linear approximation of the system (2.4) is uniformly asymptotically stable, i.e. the following system is asymptotically stable

$$\frac{dz_i(t)}{dt} = \sum_{s=1}^{m} r_{is}(t) z_s(t) + \sum_{l=1}^{m} \tau_{il}(t) z_l(t-\theta) \tag{2.5}$$

where

$$r_{is}(t) = \left[\frac{\partial X_i}{\partial x_s} + \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k}\frac{\partial f_k}{\partial x_s}\right]_{(x_s = x_s(t))}$$

$$\tau_{il}(t) = \left[\frac{\partial X_i}{\partial x_l(t-\theta)} + \sum_{k=1}^{n} \frac{\partial X_i}{\partial y_k}\frac{\partial f_k}{\partial x_l(t-\theta)}\right]_{(x_l=x_l(t))} \tag{2.6}$$

We will consider also the system of differential equations

$$\frac{dy_j}{dt} = Y_j(\alpha_s, y_k, \gamma_l, \beta) \tag{2.7}$$

where $\alpha_s$ and $\gamma_l$ have taken the places of $x_s(t)$, and $x_l(t-\theta)$, while $\beta$ replaces $t$.

Suppose that for all fixed values

$$\alpha_s = x_s(\beta), \quad \gamma_l = x_l(\beta-\theta), \quad |x_s| < \infty, \quad \beta = t, \quad t_0 \leqslant \beta < \infty$$

one can find for the system (2.7) a constant symmetric matrix $A(\alpha_s, \gamma_l, \beta)$ which is uniformly bounded in $\alpha_s$, $\gamma_l$ and $\beta$, is positive definite, and is such that the symmetrized matrix

$$\{B\}_{jk} = \left(\left\{A\frac{\partial Y}{\partial y}\right\}_{jk} + \left\{A\frac{\partial Y}{\partial y}\right\}_{kj}\right) \qquad \left(\left\{\frac{\partial Y}{\partial y}\right\}_{jk} = \frac{\partial Y_j}{\partial y_k}\right) \tag{2.8}$$

has negative characteristic numbers $r_j$ that satisfy the inequality

$$r_j < -\gamma \quad \text{при } |y_j| < \infty \quad (\gamma = \text{const} > 0)$$

Under these conditions, any solution of the system (2.7) will be asymptotically stable (see, for example, [4 (p.313)]) for arbitrary initial conditions $(y_{j0})$.

*Theorem 2.1.* Let the following conditions be satisfied for the system of differential Equations (2.1).

1) The system of Equations (2.5) is uniformly asymptotically stable.

2) For every set of fixed values $\alpha_s$, $\gamma_l$ and $\beta$, one can give for the system of Equations (2.7) symmetric matrices $A(\alpha_s, \gamma_l, \beta)$ which are uniformly bounded in $\alpha_s$, $\gamma_l$, $\beta$ and are such that the symmetrized matrix $\{B\}_{jk}$ (2.8) has negative characteristic values satisfying the inequality $r_j < -\gamma$ ($\gamma$ = const > 0). Then, for small enough values of the parameter $\mu$, the solution $x_i(t, \mu)$, $y_j(t\ \ \mu)$ of the system of Equations (2.1) will be uniformly asymptotically stable relative to small deviations $x_{i0}$ and arbitrary deviations $y_{j0}$; for any given numbers $Q > 0$, $\epsilon > 0$ there exists a number $\mu_0 > 0$ such that the next inequalities hold

$$|x_i(t,\mu) - x_i(t)| < \varepsilon, \quad |y_j(t,\mu) - y_j(t)| < \varepsilon \text{ when } t > t_1(Q, \varepsilon)$$
$$|y_j(t_0,\mu) - y_j(t_0)| < Q \tag{2.9}$$

provided $\mu < \mu_0$. Hereby $\mu_0$ may be chosen so small that $t_1$, of condition

(2.9), may differ from $t_0$ by less than any previously given number.

*Proof.* We will express the solution of the original system (2.1) in terms of the solution of the degenerate system (2.2) by means of the equations

$$\xi_i(t, \mu) = x_i(t, \mu) - x_i(t), \qquad \eta_j(t, \mu) = y_j(t, \mu) - f_j[x_s(t, \mu), x_l(t-\theta, \mu), t]$$

and the use of the equalities

$$\frac{d\xi_i(t, \mu)}{dt} = \frac{dx_i(t, \mu)}{dt} - \frac{dx_i(t)}{dt}$$

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{dy_j(t, \mu)}{dt} - \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} \frac{dx_s(t, \mu)}{dt} - \sum_{l=1}^{m} \frac{\partial f_j}{\partial x_l(t-\theta)} \frac{dx_l(t-\theta, \mu)}{dt} - \frac{\partial f_j}{\partial t}$$

Let us construct the system of differential equations of the disturbed motion separately:

when $t_0 \leqslant t \leqslant t_0 + \theta$ (2.10)

$$\frac{d\xi_i(t, \mu)}{dt} = \sum_{s=1}^{m} r_{is}(t)\, \xi_s(t, \mu) + \sum_{l=1}^{m} \tau_{il}(t)\, \xi_l(t-\theta, \mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k} \eta_k(t, \mu) + R_i(\xi_s)$$

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{Y_j\{\alpha_s^*(t), f_k[\alpha_s^*(t), \alpha_l^*(t-\theta), t] + \eta_k(t, \mu), \alpha_l^*(t-\theta), t\}}{\mu} -$$

$$- \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} \Big[\sum_{l=1}^{m} r_{sl}(t)\, \xi_l(t, \mu) + \sum_{l=1}^{m} \tau_{sl}(t)\, \xi_l(t-\theta, \mu)\Big] - \sum_{k=1}^{n} H_{jk} \eta_k(t, \mu) +$$

$$+ R_j(\xi_s) + R^*(t) - \sum_{l=1}^{m} \frac{\partial f_j}{\partial x_l(t-\theta)} \frac{dg_{l0}(t)}{dt}$$

when $t > t_0 + \theta$ (2.11)

$$\frac{d\xi_i(t, \mu)}{dt} = \sum_{s=1}^{m} r_{is}(t)\, \xi_s(t, \mu) + \sum_{l=1}^{m} \tau_{il}(t)\, \xi_l(t-\theta, \mu) + \sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k} \eta_k(t, \mu) + R_i(\xi_s)$$

$$\frac{d\eta_j(t, \mu)}{dt} = \frac{Y_j\{\alpha_s^*(t), f_k[\alpha_s^*(t), \alpha_l^*(t-\theta), t] + \eta_k(t, \mu), \alpha_l^*(t-\theta), t\}}{\mu} -$$

$$- \sum_{s=1}^{m} \frac{\partial f_j}{\partial x_s} \Big[\sum_{l=1}^{m} r_{sl}(t)\, \xi_l(t, \mu) + \sum_{l=1}^{m} \tau_{sl}(t)\, \xi_l(t-\theta, \mu)\Big] -$$

$$- \sum_{l=1}^{m} \frac{\partial f_j}{\partial x_l(t-\theta)} \Big[\sum_{s=1}^{m} r_{ls}(t)\, \xi_s(t-\theta, \mu) + \sum_{s=1}^{m} \tau_{ls}(t)\, \xi_s(t-2\theta, \mu)\Big] -$$

$$-\sum_{k=1}^{n} H_{jk}\eta_k(t, \mu) - \sum_{k=1}^{n} H_{jk}^*\eta_k(t-\theta, \mu) + R_j(\xi_s) + R_j^*(t)$$

In (2.10) and (2.11), the symbols $R_i(\xi_s)$ stand for the terms that contain $\xi_l(t, \mu)$, $\xi_s(t-\theta, \mu)$ and $\xi_l(t-2\theta, \mu)$ and their linear combinations raised to degree two or higher; the expressions

$$\sum_{k=1}^{n} \frac{\partial X_i^*}{\partial y_k}\eta_k(t, \mu), \qquad \sum_{k=1}^{n} H_{jk}\eta_k(t, \mu), \qquad \sum_{k=1}^{n} H_{jk}^*\eta_k(t-\theta, \mu)$$

represent increments of the function $X_i[x_s, y_k, x_l(t-\theta), t]$ with respect to $y_k$, expressed according to the theorem on finite differences; finally, $a^*(t) = x_s(t) + \xi_s(t, \mu)$.

By the hypotheses of the theorem, the system (2.5) is uniformly asymptotically stable. Hence, just as above in Section 1, there exists for it a positive definite functional $v[\xi_1(\vartheta), \xi_2(\vartheta), \ldots, \xi_m(\vartheta), t] = v[\xi_i(\vartheta), t]$ which satisfies the conditions (1.12) to (1.14).

In the second condition of the theorem it is assumed that for any set of fixed values

$$\alpha_s = x_s(t) = x_s(\beta), \qquad \gamma_l = x_l(t-\theta) = x_l(\beta-\theta), \qquad \beta = t \qquad (t_0 \leqslant \beta < \infty)$$

there exist for the system (2.7) symmetric matrices $A(a_s, \gamma_l, \beta)$ which are uniformly bounded in $a_s$, $\gamma_l$ and $\beta$, and which have positive characteristic values satisfying the Equations (2.8).

It is hereby assumed that the matrix $\{B\}_{jk}$ has negative characteristic values that satisfy the condition $r_j < -\gamma$ ($\gamma$ = const > 0). Hence, for the system (2.7) there exists a positive definite Liapunov function

$$w(\alpha_s, \gamma_l, \beta, \eta_k) = w(\alpha_s, \gamma_l, \beta, \eta_1, \eta_2, \ldots, \eta_n), \quad \eta_k = y_k - f_k(\alpha_s, \gamma_l, \beta)$$

whose total derivative, with $a_s$, $\gamma_l$ and $\beta$ constant, computed on the basis of (2.7), will satisfy the inequalities (16.22) of [3 (Section 16)].

Let us construct the positive definite functional of the form

$$u[t, \xi_i(\vartheta), \eta_j] = v[\xi_i(\vartheta), t] + w[x_s(t), x_l(t-\theta), t, \eta_j] \qquad (2.12)$$

One can show that the total derivatives of this functional $du[t, \xi_i(\vartheta), t]/dt$, computed with the aid of the system of equations of the disturbed motion (2.10) and (2.11) for sufficiently small values of $\mu$, will be negative in the region

$$|\xi_i|>M, \qquad |\eta_j|>M, \qquad M>0$$

We will not write out explicitly the expressions for the derivatives; we note only that although the derivatives $du/dt$, computed on the basis of the equations of the disturbed motion (2.10) and (2.11), are of a somewhat more complicated form than in the linear case treated above, nevertheless, one can carry out the arguments on the estimate of $du/dt$ in the same way as before (Section 1). Even though one has to consider here functions which are not quadratic forms, the inequalities which are characteristic for quadratic forms still remain valid. Thus we come to the conclusion of the negative definiteness of $du/dt$ outside the region $|\xi_i| > M$, $|\eta_j| > M$ for small enough values of the parameter $\mu > 0$.

Furthermore, just as for systems of linear equations, one can prove that the region $|\xi_i| > M$, $|\eta_j| > M$, $M > 0$ tends to zero when $\mu$ goes to zero. This helps to convince us of the correctness of the formulated theorem.

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